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Short Communication

On resonant rotation of a weakly damped pendulum

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Abstract

The minimum, sinusoidal drive for resonant rotation of a weakly damped pendulum and the contiguous loci of stable states in a frequency-energy plane are determined by perturbing the solution for undamped, unforced oscillations and invoking the method of harmonic balance. Instability occurs through turning-point and period-doubling bifurcations, and the resonant states are stable only in rather small frequency intervals between these bifurcations.

1. Introduction

Sinusoidly forced, periodic rotation of a pendulum of length l is governed by the nonlinear differential equation

$$\ddot{\theta} + 2\delta\dot{\theta} + \sin\theta = \varepsilon \sin \omega t, \tag{1}$$

where θ is the angular displacement from the downward vertical, $\dot{\theta} \equiv \mathrm{d}\theta/\mathrm{d}t$ is the angular velocity, δ ($\equiv 1/Q$ in Ref. [1]) is the damping ratio, ε is the ratio of the maximum driving moment to the maximum gravitational moment, ω is the driving frequency, and the unit of time is $(l/g)^{1/2}$.

Numerical solutions of Eq. (1) for $\delta = \frac{1}{8}$ have been obtained through analog simulation by D'Humieres et al. [1], and some of the present results have been reported in Ref. [2], but the analytical solution does not appear to have been published. The present paper considers the

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analytical solution for $0<\delta\ll 1$ with $\varepsilon = O(\delta)$ and shows that the minimum drive for stable rotational oscillations is given by $\varepsilon = 4.60\delta$.

2. Free oscillations

The solution of Eq. (1) for free ($\delta = \varepsilon = 0$) rotational oscillations is given by

$$\theta = 2\mathrm{am}(t/k) = \omega_0 t + 2\sum_{m=1}^{\infty} m^{-1}\mathrm{sech}\,(m\pi K'/K)\,\sin(m\omega_0 t), \tag{2a}$$

$$\omega_0 = \pi/kK, \tag{2b}$$

where am is Jacobi's amplitude of modulus k (to be determined), $K = K(k)$ is a complete elliptic integral of the first kind, $K' \equiv K(k')$, and $k' \equiv (1-k^2)^{1/2}$, all in the notation of Byrd and Friedman [3].

3. Harmonic-balance approximation

The solution for free oscillations (Section 2) suggests the two-parameter perturbation

$$\theta(\tau) = 2\mathrm{am}\,\tau, \quad \tau = \frac{\omega t - \phi}{k\omega_0} \quad (0<\delta,\varepsilon\ll 1), \tag{3a,b}$$

in which the parameters k and ϕ may be approximated by the method of harmonic balance. Substituting Eqs. (3a,b) into Eq. (1) and invoking $\dot\theta = 2(\omega/k\omega_0)\,\mathrm{dn}\tau$, $\ddot\theta = -2(\omega/\omega_0)^2\mathrm{sn}\,\tau\,\mathrm{cn}\,\tau$, and $\sin\theta = 2\sin(\mathrm{am}\,\tau)\cos(\mathrm{am}\,\tau) = 2\,\mathrm{sn}\,\tau\,\mathrm{cn}\,\tau$, where sn, cn and dn are Jacobi elliptic functions [3], yields

$$\left(1-\frac{\omega^2}{\omega_0^2}\right)\mathrm{sn}\,\tau\,\mathrm{cn}\,\tau + 2\delta(\omega/k\omega_0)\,\mathrm{dn}\,\tau = \tfrac{1}{2}\varepsilon\sin(k\omega_0\tau+\phi). \tag{4}$$

The sin 2θ — and $\mathrm{d}\theta/\mathrm{d}\tau$ — weighted averages (indicated by $\langle\rangle$) of Eq. (4) yield

$$\left(1-\frac{\omega^2}{\omega_0^2}\right)\langle\mathrm{sn}^2\tau\,\mathrm{dn}^2\tau\rangle = \tfrac{1}{2}\varepsilon\cos\phi\langle\mathrm{sn}\,\tau\,\mathrm{cn}\,\tau\sin(\pi\tau/K)\rangle \tag{5a}$$

and

$$2\delta\left(\frac{\omega}{k\omega_0}\right)\langle\mathrm{dn}^2\tau\rangle = \tfrac{1}{2}\varepsilon\sin\phi\langle\mathrm{dn}\tau\cos(\pi\tau/K)\rangle \tag{5b}$$

or, after evaluating the integrals [3],

$$\frac{\varepsilon\omega_0\cos\phi}{\omega_0^2-\omega^2} = \frac{2K}{\pi^2 k}\left(\frac{E^2-k'^2K^2}{EK'+E'K-KK'}\right)\frac{\cosh^2(\pi K'/K)}{\sinh(\pi K'/K)} \equiv L(\omega_0) \tag{6a}$$

and

$$\frac{\varepsilon\omega_0 \sin\phi}{2\delta\omega} = \frac{4E}{\pi k}\cosh\left(\frac{\pi K'}{K}\right) \equiv R(\omega_0). \tag{6b}$$

A preliminary exploration reveals that Eqs. (6a,b) yield real values of ω for $\varepsilon \ll 1$ only if $k'^2 \equiv 1 - k^2 \ll 1$, which permits the approximations

$$L \sim (4/\pi^2\omega_0)\cosh(\tfrac{1}{2}\pi\omega_0)\coth(\tfrac{1}{2}\pi\omega_0) \tag{7a}$$

and

$$R \sim (4/\pi)\cosh(\tfrac{1}{2}\pi\omega_0). \tag{7b}$$

The values of R given by Eq. (6b) and approximation (7b) are compared in Fig. 1. Substituting Eq. (7) into Eq. (6) and eliminating ϕ, yields

$$\frac{\omega^2}{\omega_0^2} = 1 - 2\delta^2T^2 \pm \left[\left(\frac{\varepsilon^2}{R^2} - 4\delta^2\right)T^2 + 4(\delta T)^4\right]^{1/2}, \tag{8a}$$

where

$$T = \pi\tanh(\tfrac{1}{2}\pi\omega_0). \tag{8b}$$

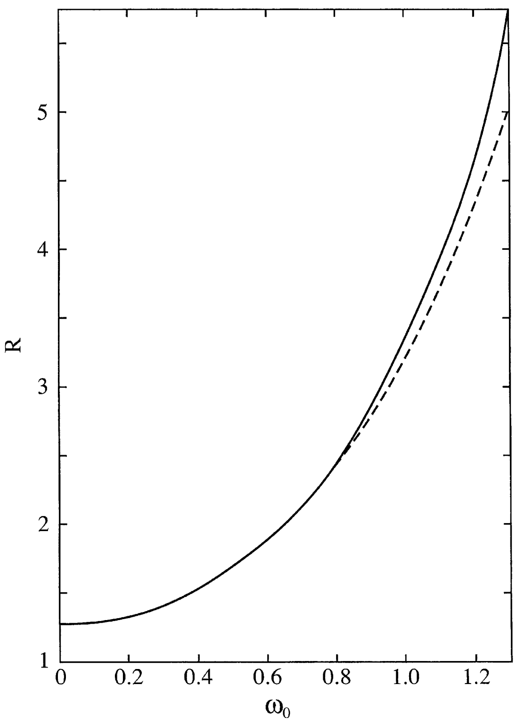

Fig. 1. $R(\omega_0)$, as given by Eq. (6b) (——) and (7b) (- - -).

The corresponding approximation to the Fourier expansion of Eq. (3a) is [cf. Eq. (2a)]

$$\theta = \omega t - \phi + 2\sum_{m=1}^{\infty} m^{-1}\operatorname{sech}\left(\tfrac{1}{2}m\pi\omega_0\right)\sin\left[m(\omega t - \phi)\right]. \tag{9}$$

It follows from Eqs. (7b) and (8) that ω^2 is real, and hence that rotational oscillations are possible (but not necessarily stable), if and only if

$$\varepsilon > (8\delta/\pi)\left[\cosh^2\left(\tfrac{1}{2}\pi\omega_0\right) - \pi^2\delta^2\sinh^2\left(\tfrac{1}{2}\pi\omega_0\right)\right]^{1/2} > 8\delta/\pi. \tag{10}$$

The dimensionless mean energy implied by Eq. (3) is

$$\langle \mathsf{E}\rangle = \langle \tfrac{1}{2}\dot{\theta}^2 + 1 - \cos\theta\rangle = \frac{2}{k^2}\left[1 + \frac{E}{K}\left(\frac{\omega^2}{\omega_0^2} - 1\right)\right] \tag{11a}$$

$$\sim 2\left[1 + 16\exp\left(-\frac{2\pi}{\omega_0}\right) + \frac{\omega_0}{\pi}\left(\frac{\omega^2}{\omega_0^2} - 1\right)\right] \quad (k \uparrow 1). \tag{11b}$$

The locus of $2K$-periodic (in τ) solutions in an $\omega - \langle \mathsf{E}\rangle$ plane (see Fig. 2) is a loop (the *resonance curve*) that lies above/below the locus of free oscillations, $\langle \mathsf{E}_0\rangle = 2/k^2$, for $\phi \gtrless \frac{1}{2}\pi$ and intersects $\langle \mathsf{E}_0\rangle$ at $\omega = 0$ and at $\omega = \omega_0 = \omega_*(\delta, \varepsilon)$, where ω_* is determined by

$$R(\omega_*) = (\varepsilon/2\delta). \tag{12}$$

It follows from the expansion about $\omega = \omega_*$, that the turning point, $\omega = \omega_1(\delta, \varepsilon)$, at which $\mathrm{d}\langle \mathsf{E}\rangle/\mathrm{d}\omega = 0$, is given by

$$\omega_1 = \omega_* + \delta^2 T_*\left(1 - \tfrac{1}{2}T_*\omega_*\right)^2 + O(\delta^4) \tag{13}$$

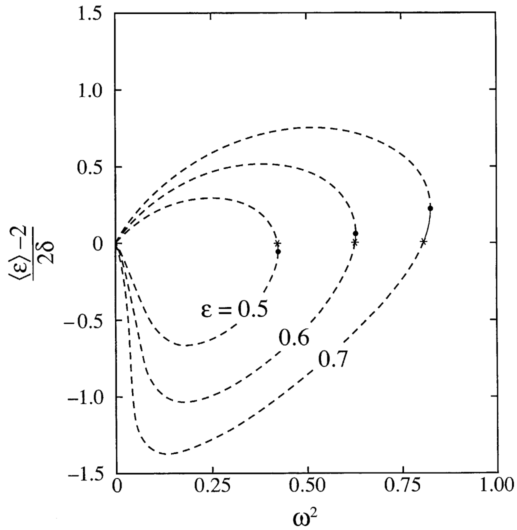

Fig. 2. The perturbation energy, as calculated from Eqs. (8a) and (11b), for $\delta = \frac{1}{8}$ and $\varepsilon = 0.5$, 0.6 and 0.7. The solid/dashed segments comprise stable/unstable states. The dots and crosses mark the turning-point ($\omega = \omega_1$) and period-doubling ($\omega = \omega_2$) bifurcation determined by Eqs. (13) and (17), respectively.

and lies on the upper branch (upper sign in Eq. (8)) of the resonance curve. The $2K$-periodic solution approximated by Eq. (3) loses stability to a $2K$-periodic perturbation at the turning point, and the remaining states on the upper branch of the resonance curve (above and to the left of the turning point) are unstable.

4. Period doubling

The $2K$-periodic solution approximated by Eq. (3) also loses stability to a $4K$-periodic solution through a period-doubling bifurcation at $\omega = \omega_2(\delta, \varepsilon)$. The solution of Eq. (1) near this period-doubling bifurcation may be approximated by

$$\theta(\tau) = 2\,\mathrm{am}\,\tau + A(t)P(\tau), \quad |A| \ll 1, \tag{14a,b}$$

$$\dot{A}(t) = O(\varepsilon A), \quad P(\tau + 4K) = P(\tau), \tag{14c,d}$$

where A is (by hypothesis) a small, slowly varying amplitude, and P is $4K$-periodic. Substituting Eq. (14a) into Eq. (1), linearizing in A, taking the moment of the result with respect to P, integrating $PP_{\tau\tau}$ by parts, and invoking Eq. (14d) yields

$$\langle P^2\rangle(\ddot{A} + 2\delta\dot{A}) + \langle(1 - 2\mathrm{sn}^2\,\tau)P^2 - (\omega/k\omega_0)^2 P_\tau^2\rangle A = 0. \tag{15}$$

It follows from Eq. (15) that a necessary condition for stability with respect to the period-doubling perturbation is

$$\langle(1 - 2\mathrm{sn}^2\,\tau)P^2 - (\omega/k\omega_0)^2 P_\tau^2\rangle \equiv \lambda(P) > 0. \tag{16}$$

The expansion of the solution of Eq. (1) in powers of ε leads to a sequence of Lamé equations of degree 1, which suggests that P be expanded in the $4K$-periodic Lamé functions $Ec_1^{2m+1}(\tau)$ and $Es_1^{2m+1}(\tau)$. The dominant member of this set is $Ec_1^1 = \mathrm{cn}\tau$ [4], which, together with the fact that the functional $\lambda(P)$ is stationary, within $1 + O(\delta^2)$, with respect to variations of P about its true value (this follows from a Lagrangian formulation), suggests the trial function $P = \mathrm{cn}\,\tau$. The resulting approximation to ω_2^2 is

$$\left(\frac{\omega_2}{\omega_0}\right) = 1 - \frac{3k'^2(E - k'^2K)}{(1 - 2k'^2)E + k'^2K} \sim 1 - 48\exp\left(-\frac{2\pi}{\omega_0}\right). \tag{17}$$

Expanding about $\omega = \omega_*$, yields

$$\omega_2 = \omega_* + 48T_*^{-1}(1 - \tfrac{1}{2}T_*\omega_*)\exp(-2\pi/\omega_*). \tag{18}$$

If $T_*\omega_* > 2$ the period-doubling bifurcation lies below the turning point, and that segment of the resonance curve between these two bifurcations comprises stable states; their vertical order is reversed, and there are no stable states, if $T_*\omega_* < 2$. It follows that $\varepsilon > 4.60\delta$ is necessary for stability; cf. Eq. (10), which implies $\varepsilon > 2.55\delta$ for the *existence* of rotational oscillations.

D'Humieres et al. [1, Figs. 1 and 3] report only a very narrow ω–ε band of rotational oscillations for $\delta = \frac{1}{8}$ and $\varepsilon < 1$, with a minimum ε of roughly 0.7 ($\varepsilon/\delta > 5.6$). A repetition of their numerical integration (but on a digital, rather than an analog, computer) confirmed their determination of a rotational state for $\delta = \frac{1}{8}$, $\varepsilon = 0.7$ and $\omega = 0.55$ and yielded $\langle \mathsf{E}\rangle = 1.88$, which compares well with $\langle \mathsf{E} \rangle = 1.79$ from Eqs. (8b) and (10). A series of runs for $\delta = \frac{1}{8}$, $\varepsilon = \frac{5}{8}$ and $\omega = 0.1(0.05)\,0.85$, with initial conditions to match Eq. (3), produced (for $\omega t \gg 1$) only swinging oscillations.

5. Subharmonic resonance

Solution (3) describes subharmonic resonance between the input and the mth harmonic in Eq. (2a) if ω_0 is replaced by $\omega = m\omega_0$ (so that $\langle \dot{\theta} \rangle = \omega/m$) in Eq. (3b). The input-dissipation balance then yields $m\pi K'/K$ in place of $\pi K'/K$ in Eqs. (6a,b) and ω in place of ω_0 in Eqs. (7a,b). Odd subharmonics do not appear in the numerical experiments of D'Humieres et al. [1], at least for $\varepsilon < 1$ (D'Humieres et al. do report oscillations for which $\langle \dot{\theta} \rangle = m\omega$ and $m\omega/n$, $m > n$, but only for $\varepsilon > 1$).

Acknowledgements

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